Kodaira dimension and hyperbolicity of families of varieties / $\mathbb{C}$

Curve C $\quad g=0 : \kappa = -\infty$
$\quad g=1 : \kappa = 0$
$\quad g \geq 2 : \kappa = 1$

Surfaces S $\quad$ ruled $: \kappa = -\infty$
$(\mathbb{Q}-)$ CY $: \kappa = 0$
Elliptic surface $: \kappa = 1$
general type $: \kappa = 2$

§1. Introduction

S: proj smooth surface $\qquad C$: proj smooth curve

$f : S \to C$ w. connected fibers

Q. Existence ?

Itaka's conj $\quad \kappa(S) \geq \kappa(C) + \kappa(F)$

F $\leftarrow$ general fiber of f

$\kappa(X)$: Kodaira dimension of a variety X

: birational invariant $\in \{-\infty, 0, 1, \cdots, \dim X\}$

$(h^0(NK_X) \sim c \cdot N^{\kappa(X)})$

Example $\quad$ If S: Calabi-Yau (or $\kappa(S) = 0$)

then $g(C) = 0$ or 1

Q. How many singular fibers are there?

Consider $\quad C = \mathbb{P}^1$ or E (elliptic curve)

Example $\quad S \to \mathbb{P}^1$ smooth

- $C_0 \times \mathbb{P}^1 \to \mathbb{P}^1$
- Hirzebruch surface $\mathbb{F}_n = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(n)) \to \mathbb{P}^1$

S is ruled (covered by $\mathbb{P}^1$) $\iff$ $\kappa(S) = -\infty$

$\iff$ K_S is not $\mathbb{Q}$-effective

Example $\quad S \to E$ smooth

- $C_0 \times E \to E$

S is not of general type $\iff$ K_S is not big

Defn $\quad$ Line bundle L: $\mathbb{Q}$-effective if $L^{\otimes m}$ has a section $\exists m >> 0$

L : big $\qquad$ if $X \overset{|mL|}{\dashrightarrow} \mathbb{P}|mL|$ $\exists m >> 0$
is generically finite

"positivity of a line bundle"

Results on # of singular fibers [surfaces : Parshin, Arakelov
3-folds / more : Kovács, Migliorini
Bedulev - Viehweg, Viehweg - Zuo]

X : smooth proj var. $\omega_X := \det(\Omega_X) = \mathcal{O}_X(K_X)$

" Canonical divisor K_X "

· $f : X \to \mathbb{P}^1$ $\Leftrightarrow \kappa(X) \geq 0$

· K_X : $\mathbb{Q}$-effective $\Rightarrow$ f has at least 3 singular fibers

$\Leftrightarrow K_{\mathbb{P}^1} + \Delta(f)$ is big

· $f : X \to E$ $\Leftrightarrow$ X is of general type

· K_X : big $\Rightarrow$ f has " 1 "

$\Leftarrow K_E + \Delta(f)$ is big

Denote $\Delta(f)$: discriminant locus

i.e. points on the base over which f is not smooth

$C \subset S \to \mathbb{P}^1 \Rightarrow$ has at least 3 singular fibers.

§2. Smooth descent of positivity of log canonical divisor

Roughly $f : X^\circ \to Y^\circ$ smooth proj morphism

of quasi-proj manifolds

$\Rightarrow$ positivity of log canonical divisor of X° descends to

" " " Y°

Setup (X, E), (Y, D) proj log smooth pairs

i.e. X : smooth proj

E : reduced snc divisor

Thm (P '22) $f : (X, E) \to (Y, D)$ a surj morphism

s.t. $E = f^{-1}D$, $f|_{X \setminus E}$: smooth

① If $K_X + E$ big, then $K_Y + D$ big

" effective, " pseudo-effective

② If $\exists \varepsilon > 0$ s.t. $K_X + (1-\varepsilon)E$ is effective

then $\exists \delta > 0$ s.t. $K_Y + (1-\delta)D$ is pseudo-effective

(Recall pseudo-effective divisor

= limit of effective divisors in $\underbrace{N^1(Y)_{\mathbb{R}}}_{\text{Neron-Severi gp}} \subset H^{1,1}(Y, \mathbb{C})$)

Rmk Compact case (no E, D) : Result of Popa-Schnell '22.

Defn A smooth quasi-proj variety V is of log general type if $K_{\bar V}+D$ is big ($V \hookrightarrow \bar V$ compactification, $D = \bar V \setminus V$: reduced snc divisor)

Cor (P'22) $f : X \to Y$ surj morphism of proj manifolds.

If either

✓ ① K_X : big or

✓ ② K_X : effective, $-K_Y$: big

Then $Y \setminus \Delta(f)$ is of log general type

(sketch of the proof)

Reduce to $f : (X, E) \to (Y, D)$

① K_X : big $\Rightarrow K_X + E$: big $\Rightarrow K_Y + D$: big

② K_X : effective $\Rightarrow K_X + (1-\varepsilon)E$: effective

$\Rightarrow K_Y + (1-\delta)D$: pseudo-effective

$\Rightarrow K_Y + D$: big $\square$

Back to # of singular fibers

$X \to \mathbb{P}^1$ K_X : effective $\Rightarrow$ at least 3 singular fibers

$X \to E$ K_X : big $\Rightarrow$ at least 1 "

Cor (P'22) $f : X \to \mathbb{P}^n$ surj. $\kappa(X) \geq 0$

Then $\dim \Delta(f) = n-1$, $\deg \Delta(f) \geq n+2$

Rmk This inequality is sharp

In particular X : proj HK manifold of dim $2n$

$X \xrightarrow{f} \mathbb{P}^n$ lagrangian fibration

$\Rightarrow \Delta(f)$ is a hypersurface of degree $\geq n+2$

∟ known Hwang-Oguiso

§3. Superadditivity of log Kodaira dimension

Defn X : smooth quasi-proj variety

$$\bar\kappa(X) := \kappa(\bar X, K_{\bar X} + D) \in \{-\infty, 0, 1, \cdots, \dim X\}$$

" log Kodaira dimension of X "

$$(h^0(N(K_{\bar X}+D)) \sim c N^{\bar\kappa(X)})$$

$\bar{\kappa}(X) = \dim X \iff X$ is of log general type.

Rmk $\bar{\kappa}(X)$: independent of compactification.

Popa's conj X, Y smooth quasi-proj varieties

$f : X \to Y$ smooth proj morphism with connected fibers.

Then $\bar{\kappa}(X) \leq \bar{\kappa}(Y) + \kappa(F)$

$\leftarrow$ fiber of f

Notice compactify $f : (\bar{X}, E) \to (\bar{Y}, D)$

- $\bar{\kappa}(X) \geq 0 \Rightarrow \bar{\kappa}(Y) \geq 0$

  $\because K_{\bar{X}} + E$ effective $\Rightarrow K_{\bar{Y}} + D$ effective

- $\bar{\kappa}(X) = \dim X \Rightarrow \bar{\kappa}(Y) = \dim Y$

  $\because K_{\bar{X}} + E$ big $\Rightarrow K_{\bar{Y}} + D$ big

Rmk Known when · Y compactifies to an Abelian variety [Meng-Popa '21]

· F has semiample canonical bundle [Campana '22]

Thm (P'22) Popa's conjecture holds when the conjectures of LMMP is true on Y.

(· Non-vanishing conjecture on Y
· very general fiber of log Iitaka fibration of Y has a good minimal model)

In particular, if $\dim Y \leq 3$, then

$\bar{\kappa}(X) \leq \bar{\kappa}(Y) + \kappa(F)$

Equality when $\dim Y = 1$ ✓

Recall Iitaka's $C_{n,m}$ conjecture

$f : X \to Y$ w. connected fibers

$\kappa(X) \geq \kappa(Y) + \kappa(F)$

$\leftarrow$ general fiber.

$\therefore$ log version $\bar{\kappa}(X) \geq \bar{\kappa}(Y) + \kappa(F)$

Main technical tool (Viehweg's fiber product trick)

Setup For $s \geq 1$

$$\underbrace{X \times_Y X \times_Y \cdots \times_Y X}_{s\text{-times}} \to Y$$

$\cup$

$X^{(s)} \longrightarrow X^s$: $\exists!$ main component dominating Y

resolution of sing $f^{(s)}$

<u>Thm</u> (Viehweg's fiber product trick) '80

$\forall N, s > 0$ $\exists$ natural inclusion

$$\left(f^{(s)}_* \omega_{X^{(s)}/Y}^{\otimes N}\right)^{\vee\vee} \hookrightarrow \left(\bigotimes^s f_* \omega_{X/Y}^{\otimes N}\right)^{\vee\vee} \checkmark$$

In the log setting

$E = f^* D$

<u>Logarithmic fiber product trick (P'22)</u>

$f : (X, E) \to (Y, D)$ $E = f^{-1} D$ $f|_{X \setminus E}$ smooth

$$\left(\bigotimes^s f_* (\omega_X(E) / \omega_Y(D))^{\otimes N}\right)^{\vee\vee} \hookrightarrow \left(f^{(s)}_* (\omega_{X^{(s)}}(E^{(s)}) / \omega_Y(D))^{\otimes N}\right)^{\vee\vee}$$

<u>Assuming conjectures of MMP,</u>

$f : X \to Y$ smooth proj morphism of quasi-proj manifolds with connected fibers.

$\bar{\kappa}(X) = \bar{\kappa}(Y) + \kappa(F)$

- Subadditivity $\geq$: Iitaka's conj
  Known when F has a good minimal model
- Superadditivity $\leq$: Popa's conj
  Known when we assume conjectures on the base Y.

<u>Rmk</u> $f : \bar{X} \to \bar{Y} \supset V$ morphism (surj) of proj manifolds

$$\kappa(\bar{Y}) + \kappa(F) \leq \kappa(\bar{X}) \leq \bar{\kappa}(V) + \kappa(F)$$

V: smooth locus of f

Iitaka's conj / Popa's conj.

K3 $S \to \mathbb{P}^1$

$24 \rightsquigarrow$ add sing fibers up to mult $= 24$.

<u>Conj</u> Kebekus–Kovács $f : X \to Y$ smooth proj

$\bar{\kappa}(Y) \geq \mathrm{Var}(f)$

$\bar{\kappa}(X) \leq \max\{\bar{\kappa}(V), \mathrm{Var}(f)\} + \kappa(F)$